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Entanglement and Pancharatnam Phase for Two Two-Level Atoms Interacting with a Single Mode Field

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Abstract A model of two 2-level atoms interacts with a single quantized electromagnetic field is considered. We study the effect of the mean photon number and the structure of the initial states of the two atoms on the dynamics of the atomic system from the separability point of view. It is found that, if we start from a product mixed atomic state, the probability of generating long living entangled states is increased as the mean photon number increases. Starting from excited atomic system in product state, one generates a more stable entangled states with high degree of entanglement. Also, the effect of the mean photon number on atomic system prepared initially in entangled states is investigated. It is found that the entangled state generated from the initially partial entangled states are more robust than those obtained from a maximum entangled state. The Pancharatnam phase for the separable and entangled states is studied under the effect of the mean photon number and the structure of the initial state. We find that for the separable states, the collapses decrease and the amplitude of the revivals is smaller than that for the entangled state, so there are long-living entangled phases. This property give us a great chances to store safely information in entangled state.

1 Introduction

Studying the dynamics of interaction between atoms a coherent field is one of the most interesting topics which has been extensively studied. A lot of applications has been studied experimentally and theoretically, see for example [1–3]. The Jaynes-Cummings model (JCM) is the simplest non-trivial model of quantum optics. It describes the interaction of a single two-level atom with a single quantized electromagnetic field. It is simple to solve and thus study properties associated with the atom and the field analytically. This model can be realized experimentally [4]. Nowadays, new applications of the atomic systems have been appeared, quantum information [5–8] and computations [9]. The most important phenomena in the atomic systems which is needed in quantum information tasks is the entanglement. So it is of a great importance to investigate how one can generate entanglement between atoms.

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In this work, we consider a model consists of two 2-level atoms interacting with a single mode radiation field. This model has been investigated from several point of view. For example Dung et al. [10], investigated the properties of the field phase, Abdel-Aty [11], considered the Pancharatnam phase of this system in the presence of the phase shift and Stark shift. Kung et al. [12], studied the entropy evolution of the system, where they considered it as a measure of the degree of entanglement. The entanglement between the atom system and the field is studied by Zeng et al. [13], where they considered the relative entropy as a measure of the degree of entanglement.

Since we want to know how can these systems contribute in performing quantum information tasks, our *treatment* of this system will be different. First of all, we shall consider a simple case of interaction. We investigate the dynamics of the atomic system from the separability point of view. We study the effect of the mean photon number and the initial state of the atomic system. Also, we quantify the amount of entanglement contained in the state of the two atoms, by the use of the negativity as a measure [14]. Since the entangled phases are used as a carrier of information, so we study behavior of the Pancharatnam phase for the separable and entangled states

The material of this article is arranged as follows: in Sect. 2, we describe our model and study its time evolution. The separable and entangled states setting are the subject of Sects. 3 and 4, where two different initial states of the atomic system are considered, product and entangled states. We investigate the separable and the non-separable behavior of the system. For the entangled states we evaluate the amount of entanglement contained in these states. Also we evaluate the fidelity of the final state when it turns into a product state. The entangled and sparable Pancharatnam is studied in Sect. 5. Finally we discuss our result in Sect. 6.

2 Dynamics of the System

The Hamiltonian which describes a system of two 2-level atoms, each consisting of states $|e\rangle$ and $|g\rangle$ coupled to a single mode radiation field, in the rotating wave approximation is given by

$$H = \omega\left(a^\dagger a + \sum_{i=1}^{2}\sigma_z^i\right) + \sum_{i=1}^{2}[\lambda_i(a^\dagger\sigma_-^i + \sigma_+^i a)] \tag{1}$$

where $a(a^\dagger)$ is the annihilation (creation) operator of the field mode, $\sigma_\pm^i$ and σ_z^i are the atomic raising, lowering and inversion operators of the two atoms. The parameter λ_i is the atom-field coupling constant and ω, is the atomic transitions and the field mode frequency. The first term in (1) represents the free-field and the non-interacting atoms, while the second term stands for the interaction Hamiltonian H_{int}. This model has been solved analytically for some special cases [10] and for a general case [16]. For simplicity, we consider the case of identical atoms i.e. $\lambda_1 = \lambda_2$. Assume that the cavity field is initially prepared in a coherent state, $|\psi(0)\rangle_f = \sum_{n=0}^{\infty} q_n|n\rangle$ and the two atoms are in a superposition states of their ground and exited states $|\psi(0)\rangle_{1,2} = a_i|g_i\rangle + b_i|e_i\rangle$, $i = 1, 2$, where 1 stands for the first atom and 2 for the second atom. The initial state of the system is assumed to be,

$$|\psi_0\rangle = \sum_{n=0}^{\infty} q_n\left(c_0^{(1)}|g_1g_2\rangle + c_0^{(2)}|e_1g_2\rangle + c_0^{(3)}|g_1e_2\rangle + c_0^{(4)}|e_1e_2\rangle\right) \otimes |n\rangle \tag{2}$$

where

$$c_0^{(1)} = a_1 a_2, \quad c_0^{(2)} = a_2 b_1, \quad c_0^{(3)} = a_1 b_2, \quad c_0^{(4)} = b_1 b_2 \tag{3}$$

with $|a_i|^2 + |b_i|^2 = 1$ for $i = 1, 2$. At any time $t > 0$, the atoms-field state is described by the state

$$\begin{aligned} |\psi(t)\rangle &= \mathcal{U}(t)|\psi(0)\rangle \\ &= \sum_{n=0}^{\infty}\big[\big(c_n^{(1)}(t)|g_1 g_2, n+2\rangle + c_n^{(2)}(t)|e_1 g_2, n+1\rangle \\ &\quad + c_n^{(3)}(t)|g_1 e_2, n+1\rangle + c_n^{(4)}(t)|e_1 e_2, n\rangle\big)\big] \end{aligned} \tag{4}$$

where $\mathcal{U}(t) = \exp(-iH_{\text{int}})$ is a unitary operator and $|g_1, g_2, n+2\rangle$, $|e_1, g_2, n+1\rangle$, $|g_1, e_2, n+1\rangle$ and $|e_1, e_2, n\rangle$ are the eigenstates of the free-atoms field part. The coefficients $c_i(t)$, $i = 1, 2, 3, 4$ are given by

$$\begin{pmatrix} c_n^{(1)}(t) \\ c_n^{(2)}(t) \\ c_n^{(3)}(t) \\ c_n^{(4)}(t) \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ u_{21} & u_{22} & u_{23} & u_{24} \\ u_{31} & u_{32} & u_{33} & u_{34} \\ u_4 & u_{42} & u_{43} & u_{44} \end{pmatrix} \begin{pmatrix} c_0^{(1)} \\ c_0^{(2)} \\ c_0^{(3)} \\ c_0^{(4)} \end{pmatrix} \tag{5}$$

with

$$\begin{aligned} & u_{11} = \frac{1}{\mu}[\mu_n \cos(t\sqrt{\mu_n}) - 2\beta_n^2(1 - \cos(t\sqrt{\mu_n}))], \quad u_{12} = -i\frac{\gamma_n}{\sqrt{\mu_n}}\sin(t\sqrt{\mu_n}), \\ & u_{13} = -u_{12}, \quad u_{14} = \frac{-2\gamma_n\beta_n}{\mu_n}(1 - \cos(t\sqrt{\mu_n})), \quad u_{21} = u_{12}, \\ & u_{22} = \frac{1}{2}(1 - 3\cos\tau\sqrt{\mu_n}), \quad u_{23} = -\frac{1}{2}(1 - \cos(\tau\sqrt{\mu})), \quad u_{24} = i\frac{\beta_n}{\sqrt{\mu_n}}\sin(\tau\sqrt{\mu_n}), \\ & u_{31} = u_{13}, \quad u_{32} = u_{23}, \quad u_{33} = u_{22}, \quad u_{34} = -u_{24}, \\ & u_{41} = u_{14}, \quad u_{42} = u_{24}, \quad u_{43} = u_{42}, \quad u_{44} = \frac{2\gamma_n^2}{\mu_n} - \frac{2\gamma_n^2 - \mu_n}{\mu_n}\cos(\tau\sqrt{\mu}) \end{aligned} \tag{6}$$

where

$$\gamma_n = \sqrt{n+1}, \quad \beta_n = \sqrt{n+2}, \quad \mu_n = 2(\gamma_n^2 + \beta_n^2), \quad \tau = \lambda t. \tag{7}$$

From (3), (5) and (6), we can obtain the density operator of the system at any time for any initial state setting. This density operator, $\rho = |\psi\rangle\langle\psi|$, where $|\psi\rangle$ is given by (4), is $2 \otimes 2 \otimes \infty$ system. The separability of this system can not be investigated i.e.,we can not know when this state behaves as entangled or separable state. Bose et al. [17], used the projection method to evaluate the minimum amount of entanglement contained in $2 \otimes \infty$. Metwally et al. [7], have used this method to quantify the minimum amount of entanglement between the atom and the field and investigate when this entanglement is enough to perform quantum teleportation. On the other hand the separability of $2 \otimes N$ and $N \otimes M$ dimensional has been studied in [18, 19]. Also the separability of the system of $2 \otimes 2 \otimes M$ has been studied by Karans and Lewenstein [20], where the system is separable if its positive partial transpose has rank $\leq N$.

Since we are interested in studying some properties of entangled and separable state of the two atoms, we shall trace out the field from the density operator of the total system. Then the reduce density operator of the system is given by

$$\rho_{\text{atoms}} = \begin{pmatrix} |c_n^{(1)}|^2 & c_n^{(1)} c_{n+1}^{*(2)} & c_n^{(1)} c_{n+1}^{*(3)} & c_n^{(1)} c_{n+2}^{*(4)} \\ c_{n+1}^{(2)} c_n^{*(1)} & |c_n^{(2)}|^2 & c_n^{(2)} c_n^{*(3)} & c_n^{(2)} c_{n+1}^{*(4)} \\ c_{n+1}^{(3)} c_n^{*(1)} & c_n^{(3)} c_n^{*(2)} & |c_n^{(3)}|^2 & c_n^{(3)} c_{n+1}^{*(4)} \\ c_{n+2}^{(4)} c_n^{*(1)} & c_{n+1}^{(4)} c_n^{*(2)} & c_{n+1}^{(4)} c_n^{*(3)} & |c_n^{(4)}|^2 \end{pmatrix} \tag{8}$$

This state is a $2 \otimes 2$ system, one can use the positive partial transpose criterion (PPT) to study its separability [21, 22]. This state is separable if its partial transpose are non-negative. In what follows, we shall consider two different initial states namely: the two atoms are in a product state and the two atoms are in an entangled state.

3 The Two Atoms Are in a Product State

Let us assume that our system initially starts from the following initial state

$$\psi_{12}(0) = |e\rangle_1 \otimes (a_2|e\rangle_2 + b_2|g\rangle_2) \tag{9}$$

where the first atom in the exited state $|e\rangle$, while the second atom in a superposition of its exited and ground state, i.e $|\psi_2\rangle = a_2|e\rangle + b_2|g\rangle$, where $|a_2|^2 + |b_2|^2 = 1$. Using (5), (6) and (9), we can write explicitly the coefficients $c_n^{(i)}(t)$, $i = 1, 2, 3, 4$ as,

$$\begin{aligned} c_n^{(1)}(t) &= -\sum_{n=0}^{\infty} q_n^2 \frac{\gamma_n}{\sqrt{\mu_n}} \left(\frac{b_2 \beta_n}{\sqrt{\mu_n}} (1 - \cos t\sqrt{\mu_n}) + i a_2 \sin t\sqrt{\mu_n} \right), \\ c_n^{(2)}(t) &= \sum_{n=0}^{\infty} q_n^2 \left(\frac{2a_2}{\mu_n} (\beta_n^2 + \gamma_n^2) \sin^2 t\sqrt{\mu_n} + a_2 \cos t\sqrt{\mu_n} - i \frac{2\beta_n}{\sqrt{\mu_n}} \sin t\sqrt{\mu_n} \right), \\ c_n^{(3)}(t) &= -\sum_{n=0}^{\infty} q_n^2 \frac{\sin t\sqrt{\mu_n}}{\sqrt{\mu_n}} \left(\frac{2a_2}{\sqrt{\mu_n}} (\beta_n^2 + \gamma_n^2) + b_2 \beta_n \right), \\ c_n^{(4)}(t) &= \sum_{n=0}^{\infty} q_n^2 \frac{\cos t\sqrt{\mu_n}}{\sqrt{\mu_n}} \left(\frac{2b_2 \gamma_n^2}{\sqrt{\mu_n}} (\mu_n - 2\gamma_n^2) - i a_2 \beta_n \right). \end{aligned} \tag{10}$$

By using (8) and (10), one can obtain the density operator of the system at any time t. Now we want to know when the state behaves as an entangled state and when as product state.

To study the separability, we evaluate the eigenvalues of this partial transpose of the density matrix numerically. If the state is entangled there should be at least a negative eigenvalue [21, 22]. Figure 1 shows the entangled and separable behavior of the two atoms for different values of the mean photon number $\bar{n}$. We plot the smallest eigenvalues against the dimensional time, $\tau = \lambda t$. It is clear that, as soon as the interaction times goes on an entangled state is generated. Also as one increases the values of $\bar{n}$, the possibility that the two atoms are in an entangled state increases. Due to the Rabi oscillations, the state fluctuate from

Fig. 1 The negativity: the *dot curve* for $\bar{n} = 10$, the *dashed curve* for $\bar{n} = 15$ and the *solid* one for $\bar{n} = 20$

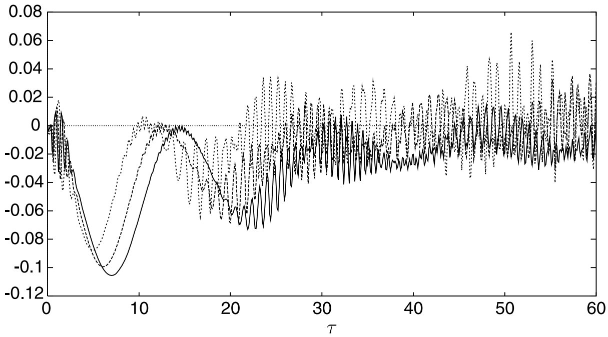
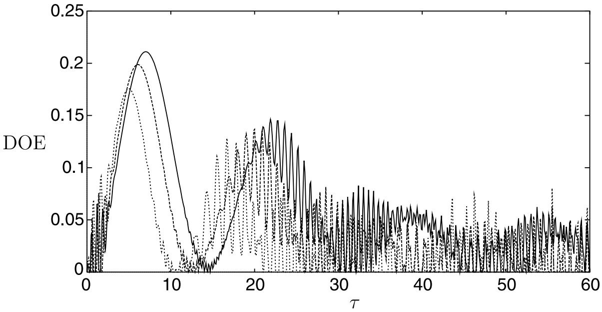


Fig. 2 The degree of entanglement as a function of the scaled time λt. The *solid curve* is drown for $\bar{n} = 10$, the *dashed curve* for $\bar{n} = 15$ while the *solid curve* for $\bar{n} = 20$



separable to entangled. The time which is taken in a separability position is decreases as $\bar{n}$ increases [8]. To quantify the amount of entanglement contained in the entangled states, we shall use a measurement introduced by K. Zyczkowski [14]. This measure states that if the eigenvalues of the partial transpose are given by μ_i, $i = 1, 2, 3, 4$, then the degree of entanglement, DOE is defined by

$$\text{DOE} = \sum_{i=1}^{4} |\mu_i| - 1. \tag{11}$$

It is clear that the DOE is equal to zero for separable states and it is equal to one for the maximal entangled states. Figure 2, shows the behavior of the DOE as a function of the scaled time τ. Three different values for the mean photon number $\bar{n}$ are considered. This figure depicts as one increases $\bar{n}$, the DOE increases and the Rabi frequency are shifted. This figure coincide with Fig. 1, where the DOE vanishes for separable state. Also for $\bar{n} = 20$, one can obtain a stable entangled state most of the time.

As we have seen the degree of entanglement depends on the structure of the initial state. We are interested in the entangled states which are generated from a product states. So when the state turn into a separable state we evaluate its fidelity, how much is related to the initial state. In this context, the fidelity of the first atom is given by,

$$F_1 = |c_n^{(2)}|^2 + |c_n^{(4)}|^2, \tag{12}$$

while for the second atom,

$$\begin{aligned} F_2 = {} & |b_2|^2 \big(|c_n^{(1)}|^2 + |c_n^{(2)}|^2\big) + a_2^* b_2 \big(C_{n+1}^{(3)} c_n^{*(1)} + C_{n+1}^{(4)} c_n^{*(2)}\big) \\ & + b_2 a_2^* \big(C_n^{(1)} c_{n+1}^{*(3)} + C_n^{(2)} c_{n+1}^{*(4)}\big) + |a_2|^2 \big(|c_n^{(3)}|^2 + |c_n^{(4)}|^2\big). \end{aligned} \tag{13}$$

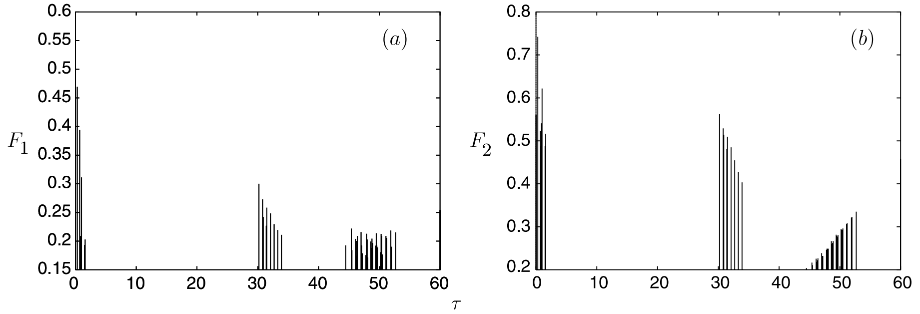
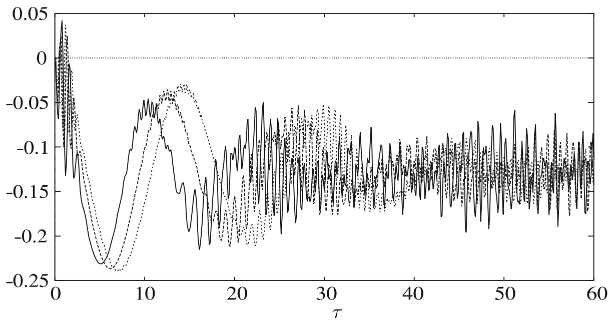


Fig. 3 The fidelity: (**a**) F_1 for the first atom, (**b**) F_2 for the second atom with $\bar{n} = 20$, $a = 0.5$ and $b = \sqrt{1 - a^2}$

Fig. 4 The same as Fig. 1, but the two atoms are initially in a product excited state



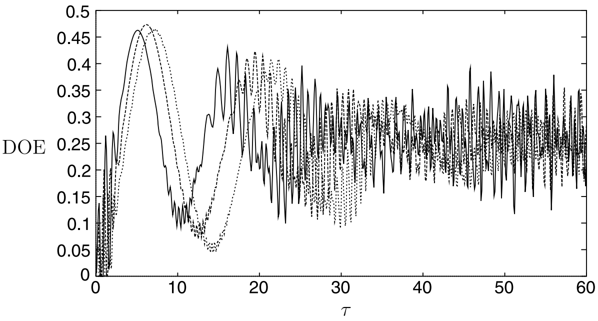
We plot these fidelities against the scaled time in Fig. 3. From this figure it is very clear that when the atomic system behaves as an entangled system and when as a product system. Also the fidelity of the second atom when turn into a separable state is much larger than the first atom. This is due to that the second atom is in a superposition while the first in its exited state. Another remark these fidelities decrease with time, this explain why the degree of entanglement decreases with time.

Finally, we end this section by the second case where we consider that the atomic system is prepared initially in an *excited* state i.e. $|\psi(0)\rangle_{ab} = |ee\rangle$. For this choice, the initial state is defined by $a_1 = a_2 = 0$ and $b_1 = b_2 = 1$ while the coefficients $c_n^{(i)}(t), i = 1, 2, 3, 4$ are defined as,

$$
\begin{aligned}
c_n^{(1)}(t) &= -\sum_n^{\infty} q_n^2 \frac{2\gamma_n \beta_n}{\mu_n}(1 - \cos(\tau\sqrt{\mu})), \\
c_n^{(2)}(t) &= \sum_n^{\infty} q_n^2 \frac{i\beta_n}{\sqrt{\mu_n}} \sin(\tau\sqrt{\mu_n}), \qquad c_n^{(3)}(t) = -c_n^{(2)}(t), \\
c_n^{(4)}(t) &= \sum_n^{\infty} q_n^2 \left(\frac{2\gamma_n^2}{\mu_n} - \frac{2\gamma_n^2 - \mu_n}{\mu_n}\cos(\tau\sqrt{\mu_n})\right).
\end{aligned}
\tag{14}
$$

Using these coefficients, one can get the density operator (8) explicitly. In Fig. 4, we plot the PPT criterion for different values of $\bar{n}$. It is clear that as soon as the interaction is switched

Fig. 5 The same as Fig. 2, but the two atoms are initially in a product excited state



on, an entangled state is generated. Then for a short time the state turns into separable state. This is due to the two excited atoms are unstable. As time goes on the state becomes entangled. The usual effect of the mean photon number $\bar{n}$ is seen where the Rabi oscillations are shifted and decrease as one increases the values of the mean photon numbers. Comparing Fig. 1 and Fig. 4, we can see that generating an entangled state by using two atoms that has been initially in a product excited states is better than the superposition case which studied. The degree of entanglement is plotted in Fig. 5, it is clear that the amount of entanglement depicted in this case is much larger than those plotted in Fig. 2. Since we start from excited states the interaction does not need a lot of energy. This explain the slightly effect of the mean photon number $\bar{n}$ on the degree of entanglement. On the other hand one does not need to evaluate the fidelity of the output state. In fact this final state does not change into a product state. So starting from a product exited state, one can get along living entangled states, which represents an important result in the context of quantum information [15].

4 The Two Atoms Are in an Entangled State

In this section, we assume that the two atoms are initially in an entangled state. We define it as,

$$|\psi(0)\rangle_{12} = a|eg\rangle + b|ge\rangle \tag{15}$$

where $|a|^2 + |b|^2 = 1$ as usual and the field in its coherent state. By using (5), (6) and (15), one gets the final state $|\psi(t)\rangle$ which is defined by (4), where the coefficient $c_n^{(i)}(t)$, $i = 1, 2, 3, 4$ are given by

$$\begin{aligned}
c_n^{(1)} &= -i\sum_{n=0}^{\infty} q_n(a+b)\frac{\gamma_n}{\sqrt{\mu_n}}\sin\tau\sqrt{\mu_n}, \\
c_n^{(2)} &= \sum_{n=0}^{\infty} q_n\left(\frac{\beta_n^2+\gamma_n^2}{\mu_n}(a-b)(1-\cos\tau\sqrt{\mu_n}) - a\cos\tau\sqrt{\mu_n}\right), \\
c_n^{(3)} &= \sum_{n=0}^{\infty} q_n\left(\frac{\beta_n^2+\gamma_n^2}{\mu_n}(b-a)(1-\cos\tau\sqrt{\mu_n}) + b\cos\tau\sqrt{\mu_n}\right), \\
c_n^{(4)} &= -i\sum_{n=0}^{\infty} q_n(a+b)\frac{\beta_n}{\sqrt{\mu_n}}\sin\tau\sqrt{\mu_n}.
\end{aligned} \tag{16}$$

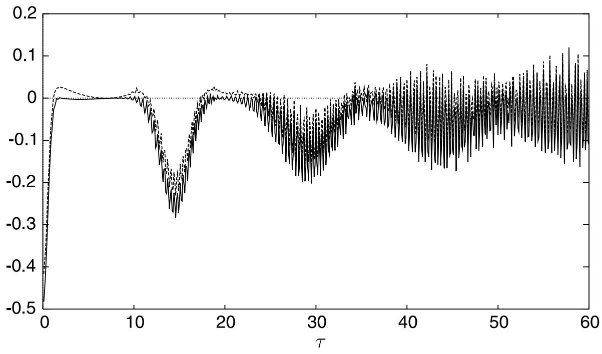


Fig. 6 The PPT criterion where the two atoms are initially: in a maximum entangled state (*dotted line*) and for partial entangled state (*solid line*) with $\bar{n} = 20$

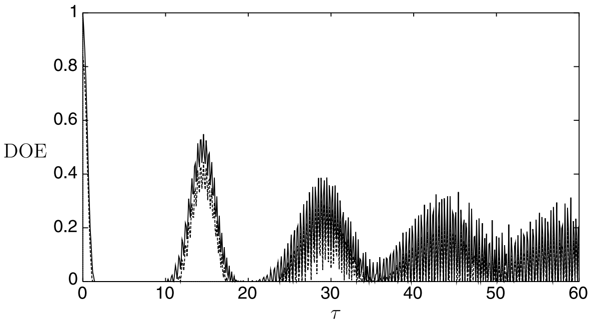


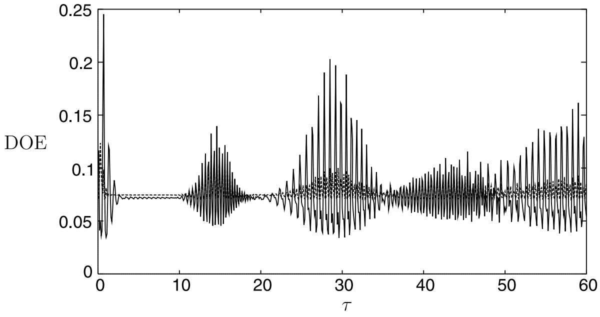
Fig. 7 The degree of entanglement evaluated for $\bar{n} = 20$. The *solid line* for the two atoms are initially in maximally entangled state, while the *dotted lines* for the two atoms are initially in partially entangled states with $a = 0.5$ and $b = \sqrt{1 - a^2}$

Now all the details are on hand to obtain the density operator (8). To study the behavior of this state from the separability point of view, one has to apply the PPT criterion. So we have to evaluate the partial transpose of the density operator (8) with $c's$ are given by (16). In Fig. 6, we compare the PPT criterion, as a function of the scaled time τ, between two cases: the first, where the two atoms are initially in maximally entangled state (MES) while the second, the two atoms are in a partially entangled state (PES). In the two cases we consider the mean photon number $\bar{n} = 20$. From this figure its clear that an entangled state is generated as soon as the interaction is switched on. Also for the MES case the state remains entangled for a long time, but in a short time it turns into a product state. For the state generated from PES, it turns quickly into a product state and as time goes on it vibrates between separable end entangled behavior. The degree of entanglement (DOE) is plotted in Fig. 7, it is clear that its behavior is similar to the previous case. Moreover the DOE for the case MES is much larger than the PES and it decreases with time. Also this figure shows that the DOE vanish for a large interval of time which is coincide with the perdition of Fig. 6. This means that the state behaves as a product state in this interval of time. We can see that there is a very fast losing of entanglement. This phenomena is called the sudden death of entanglement which means that the degree of entanglement does not decay smoothly [31–36].

Now, it is possible to evaluate the fidelity of the final state, namely we evaluate how much it is related to the initial state. In this case the fidelity is given by

$$F = |a|^2|c_n^{(2)}| + a^* b c_n^{(2)} c_n^{*(3)} + b^* a c_n^{(3)} c_n^{(2)} + |b|^2 |c_n^{(3)}|^2. \tag{17}$$

Fig. 8 The fidelity for final state, where *solid* and the *dotted lines* for a atomic system is initially in a partial and maximal entangled states respectively



In Fig. 8, we plot the fidelity of the final state for the two cases the MES and the PES. The more surprising result is that the fidelity of the final state which is generated from an atomic system is initially in a maximum entangled state is much smaller than this which obtained from PES. This means that the degree of entanglement generated by PSE is much robust than that obtained from MES.

5 Pancharatnam Phase

In the context of quantum information, phases play an important role in coding information, where they are used as carriers of quantum bits. Also due to its periodical behavior they are very important for quantum computation and communication [23, 24]. In our study, we shall consider the Pancharatnam phase which represents the total phase of the system. This phase includes two types of phases: the dynamical phase which depends on the Hamiltonian of the system, that can be evaluated easily. The other type is the geometrical phase, which depends on the path in the space spanned by all the likely quantum states of the system. So it is difficult to calculate it, but it is considered as the difference between the total phase (Pancharatnam) phase and the dynamical phase. This physical phenomena has been extensively studied for different systems experimentally [25–27] and theoretically, see for example [28–30]. In this treatment, we shall study how the Pancharatnam phase behaves for the entangled and separable states. Also we study the effect of the mean photon number and the structure of the initial state on this behavior. As we know from the literature that the Pancharatnam phase ϕ_p between the initial vector $|\psi(0)\rangle$ and the final vector $|\psi(t)\rangle$ is given by

$$\phi_p = \arg\langle\psi(0)|\psi(\tau)\rangle = -\sin^{-1}\left(\frac{Y(\tau)}{\sqrt{X(\tau)^2 + Y(\tau)^2}}\right) \tag{18}$$

where

$$\begin{aligned} X(t) = \sum_{n=1}^{\infty} q_n^2 \Bigg[\frac{\alpha b^2}{\mu_n \sqrt{n+1}} \Bigg\{ (\beta_{n+1}^2 + \gamma_{n+1}^2)(1 - \cos\tau\sqrt{(\mu_{n+1})} \\ + \mu_{n+1} \cos\tau\sqrt{\mu_{n+1}} \Bigg\} \Bigg], \end{aligned} \tag{19}$$

$$Y(t) = -\sum_{n=1}^{\infty} q_n^2 ab \left\{ \frac{\alpha\beta_{n+1}}{\sqrt{\mu_{n+1}}} \sin\tau\sqrt{\mu_{n+1}} + \frac{\beta_n}{\sqrt{\mu_n}} \sin\tau\sqrt{\mu_n} \right\}.$$

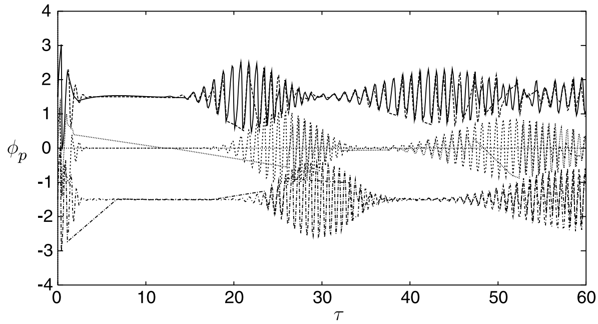


Fig. 9 The Pancharatnam phase for the entangled and separable state for the two atom state: the most upper one is $\phi_p + 1.5$ for $\bar{n} = 10$; the *solid curve* for separable state while the *dotted curve* for the entangled state. The next one is ϕ_p for $\bar{n} = 15$, where the *dashed curve* for entangled state and the *dotted line* for separable state. The most bottom one is $\phi_p - 1.5$ for $\bar{n} = 20$, the *dash-dotted curve* for separable states. For all cases we consider $a = 0.5$, $b = \sqrt{1 - a^2}$

In Fig. 9, we plot the Pancharatnam phase as a function of the scaled time τ. We consider three different values for the mean photon number, $\bar{n} = 10$, 15 and 20. As a first remark this figure depicts the collapse and revival of the ordinary Rabi oscillations. As one increases the mean photon number, the collapse increases and the amplitude decreases a little bit. As a second remark, the Pancharatnam phase doses not depend on the time for a long range and equal zero, and as we see from Fig. 2, the state is entangled most of the time in this range. So, one can use these phases to carry information or performing quantum computation. From this figure, we can see the behavior of the separable phases, i.e. it is the phase where its corresponding state is separable. We can see that the number of revival decreases, since the atoms spend most of the time in the entangled area. Also it decreases more as one increase $\bar{n}$. The amplitude of its revival is smaller comparing with entangled situation.

6 Conclusions

In this work, we treat the problem of the two-two level atoms interact with a single field from a different point of view. We study the separability problem of the two atoms under the effect of the mean photon number $\bar{n}$ and the structure of the initial state of the two atoms. We consider two classes for the atomic initial system: first, the two atoms are initially in product state while the second the two atoms are initially in an entangled state. For the *first* class, we find that if the atomic system is initially in an excited product state one can generate a long-living entangled state. The amount of entanglement contained in these state is much better than any other choice. For this case the degree of entanglement is a little bit decreases as the mean photon number increases, this is due to the thermal effect of the field. Also for this class we consider another case, the two atoms are initially in superposition product state. We find that the probability of generating a long-living entangled state increases as the mean photon number is increased. For this case, when the atomic system behaves as a separable, the fidelity of the first atom is smaller than that for the second atom. This is due to that the superposition state is more robust.

For the *second* class, we consider two cases: the atomic system is initially in maximum and partial entangled state. One can generate unstable long-living entanglement. The phenomena of the sudden death of entanglement is appear for this class. So starting from an excited product state, one can generate a more stable and robust long-living entangled state. If we start with a partial entangled state the fidelity of the final state is much better than that obtained from atomic system prepared initially in a maximum entangled state.

The behavior of the Pancharatnam phase for separable and entangled states is investigated. It is clear that the phenomena of the collapses and revivals appears. For the entangled atoms, as one increases $\bar{n}$, the collapses is shifted and the amplitude of the revivals decrease. If we consider the separable states, we can see that the collapses decrease and the amplitude of the revivals is smaller than that for the entangled state. This study is more important in the context of quantum information, where one can send and store information by using phases.

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